# Reflection Principle and Construction of Saturated Ideals on $\mathcal{P}_{\omega_1}\lambda$

Hiromi ISHII[1] [2]

University of Tsukuba

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# The Goal: Saturated Filters

⋆ In this talk, we will prove the following well-known theorem:

**Theorem 1 (Foreman–Magidor–Shelah)**

*Let $\delta$ be a supercompact cardinal, $G$ a $\mathrm{Col}(\omega_1, {<}\delta)$-generic filter over $V$. Then, in $V[G]$, there is an $\aleph_2$-saturated filter on $\omega_1$.*

- We want to understand it more clearly and see what's going on in $V^{\mathrm{Col}(\omega_1, <\delta)}$.
- We use Reflection Principle at each intermediate stage, which generalise the following standard stationary reflection:

**Theorem 2 (F.–M.–S.)**

*Let $\kappa$ be supercompact and $G$ a $\mathrm{Col}(\omega_1, {<}\kappa)$-generic over $V$. Then in $V[G]$, for any stationary $S \subseteq \mathcal{P}_{\omega_1}\mathcal{H}_\theta$ there is $A \subseteq H_\theta$ with $|A| = \aleph_1$ such that $S \cap \mathcal{P}_{\omega_1} A$ is stationary.*

# Conventions

In what follows:

- By "*normal filter*", we mean "*$\sigma$-complete normal fine filter*".
- $\delta$ denotes a supercompact cardinal.
- $E := \{ \kappa \leq \delta \mid \kappa : 2^\kappa\text{-s.c.} \}$, $I := \{ \kappa \leq \delta \mid \kappa : \text{inaccessible} \}$.
- For any $A \subseteq \mathrm{On}$ and $\alpha \in \mathrm{On}$, $\alpha^{+A} := \min \{ \beta \in A \mid \beta > \alpha \}$, i.e. the successor of $\alpha$ in $A$. In particular, we write $\bar{\alpha} := \alpha^{+I}$.
- $\mathbb{P}_\alpha := \mathrm{Col}(\omega_1, {<}\alpha)$, i.e. the Lévy collapse making $\alpha$ to be $\omega_2$.
- If $G$ is a $(V, \mathbb{P}_\delta)$-generic and $\alpha \leq \delta$, then $G_\alpha := G \cap \mathbb{P}_\alpha$.
- If we write $N \prec \mathcal{H}_\theta$, we implicitly assume $N$ to be countable.

Let $\mathcal{F}$ be a filter on $\mathcal{P}_{\omega_1} X$.

- $\mathcal{F}^* := \{ A \subseteq \mathcal{P}_{\omega_1}\lambda \mid A^c \in \mathcal{F} \}$ is called the *dual ideal* of $\mathcal{F}$.
- $\mathcal{F}^+$ denotes the collection of all *$\mathcal{F}$-positive sets*; i.e. $A \in \mathcal{F}^+$ iff $A \cap S \neq \emptyset$ for any $S \in \mathcal{F}$.
  - We regard $\mathcal{F}^+$ as a poset, ordered by the inclusion modulo $\mathcal{F}^*$.
  - We compute $\mathcal{F}^+$ in the universe where $\mathcal{F}$ is defined.

# Saturation and Generic Embeddings

Let $\mathcal{F}$ be a filter on $\mathcal{P}_{\omega_1}\lambda$.

- As stated before, we will consider the *saturation* of filters.
- $\mathcal{F}$ is *$\kappa$-saturated* if $\mathcal{F}^+$ has $\kappa$-c.c. as a forcing notion.
- We say $\mathcal{F}$ is *saturated* if it is $\lambda^+$-saturated.
- The notion of saturation is closely related to *generic ultrapower*.
  - Forcing by $\mathcal{F}^+$ adds an ultrafilter $\dot{G}$ on $\mathcal{P}^V_{\omega_1}X$ extending $\mathcal{F}$.
  
  $\rightsquigarrow$ In $V[G]$, one can consider a *generic ultrapower* $\mathrm{Ult}(V, G)$.

? When is $\mathrm{Ult}(V, G)$ well-founded?

**Fact 3 (Solovay?)**

*If $\mathcal{F}$ is saturated, then $\mathrm{Ult}(V, G)$ is always well-founded and its transitive collapse $M$ is closed under $\lambda$-sequences.*

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# Construction on $\omega_1$: An Overview

- We construct an increasing normal filters $\langle \mathcal{F}_\kappa \mid \kappa \leq \delta \rangle$, where $\mathcal{F}_\kappa = (\text{the normal closure of } \langle S_\mu \mid \mu \in E \cap \kappa \rangle)^{V[G_\kappa]}$.

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- For any m.a.c. $\mathcal{A}$ of $\mathcal{F}_\delta$, an easy closure argument shows that there are club many $\kappa < \delta$ with $\mathcal{A}_\kappa := \mathcal{A} \cap V[G_\kappa] \in V[G_\kappa]$ and $\mathcal{A}_\kappa$ is an m.a.c. of $\mathcal{F}_\kappa$ in $V[G_\kappa]$.

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$\rightsquigarrow$ $\mathcal{A} = \mathcal{A}_\kappa \in V[G_\kappa]$ for some $\kappa < \delta$.

- There are only $(2^{\aleph_1})^{V[G_\kappa]} < \delta = \aleph_2^{V[G_\delta]}$ subsets of $\omega_1$ in $V[G_\kappa]$, hence we get $|\mathcal{A}| = |\mathcal{A}_\kappa| < \aleph_2$.

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# Witnessing Maximality, I

- We use the following classical characterisation of maximality:

**Fact 4**

*Let $\mathcal{F}$ be a normal filter on $\omega_1$. Then the supremum of $\mathcal{A} = \{ A_\alpha \mid \alpha < \omega_1 \} \subseteq \mathcal{F}^+$ in $\mathcal{F}^+$ is given by the diagonal union. In particular, if $\mathcal{A}$ is an antichain then $\mathcal{A}$ is maximal if and only if $\bigtriangledown_\alpha A_\alpha \in \mathcal{F}$.*

! Note: Since we deal with Lévy collapse, every m.a.c. of $\mathcal{F}_\kappa$'s ($\kappa < \delta$) is eventually of size $\aleph_1$.

⇝ In particular, for every m.a.c. $\mathcal{A}$ of $\mathcal{F}_\kappa$, we can add stationary set witnessing $\bigtriangledown \mathcal{A} \in \mathcal{F}_{\kappa+I}$ at the stage $\kappa^{+I}$!

- As usual, we want to use elementary submodels to make argument simpler.

# Witnessing Maximality, II

- In $V[G_{\kappa^+}]$, we can project large submodels in $V[G_\kappa]$ onto $\aleph_1$ to get desired stationary set to be added.

## Definition 5

Let $\kappa < \delta$. Since, in $V[G_{\bar{\kappa}}]$, $\mathcal{H}^{(\kappa)} := \mathcal{H}_{\kappa^+}^{V[G_\kappa]}$ is of size $\aleph_1$, one can pick $\left\langle \dot{N}_\alpha^\kappa \,\middle|\, \alpha < \omega_1 \right\rangle$ such that, in $V[G_{\bar{\kappa}}]$, $\mathcal{H}^{(\kappa)} = \bigcup_\alpha \dot{N}_\alpha^\kappa$ and $\left\langle \dot{N}_\alpha^\kappa \,\middle|\, \alpha < \omega_1 \right\rangle$ is a continuous elementary $\in$-chain. Then we define, in $V[G_{\bar{\kappa}}]$, $\pi_\kappa : \mathcal{P}\mathcal{P}_{\aleph_1}\mathcal{H}^{(\kappa)} \to \mathcal{P}\omega_1$ by:

$$\pi_\kappa(\tilde{S}) := \left\{ \alpha < \omega_1 \,\middle|\, N_\alpha^\kappa \in \tilde{S} \right\}.$$

## Remark

$\tilde{S} \subseteq \mathcal{P}_{\aleph_1}\mathcal{H}^{(\kappa)}$ is stationary iff $\pi_\kappa(\tilde{S})$ is stationary in $\omega_1$.

# Witnessing Maximality, III: Indestructibility Lemma

Finally, we can state what the "stationary set witnessing maximality" is:

**Lemma 6**

*Suppose $\kappa < \delta$ be inaccessible, $\mu \geq \kappa^+$ and $\mathcal{A}$ an antichain in $\mathcal{F}_\kappa^+$. In $V[G_\kappa]$, let*

$$\tilde{S}_{\mathcal{A}} := \left\{ N \prec \mathcal{H}_{\kappa^+}^{V[G_\kappa]} \,\middle|\, \mathcal{A}, \mathcal{F}_\kappa \in N \wedge N \cap \omega_1 \in \bigcup (\mathcal{A} \cap N) \right\}.$$

*In $V[G_\mu]$, if $\mathcal{F}$ is a normal filter on $\omega_1$ extending $\mathcal{F}_\kappa$, $(\mathcal{F}_\kappa^+) \cap V[G_\kappa] \subseteq \mathcal{F}^+$ and $\pi_\kappa(\tilde{S}_{\mathcal{A}}) \in \mathcal{F}$, then $\mathcal{A}$ is maximal in $\mathcal{F}$.*

Proof

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- Instead of adding $\tilde{S}_{\mathcal{A}}$ for all relevant m.a.c.'s $\mathcal{A}$ of $\mathcal{F}_\kappa$, we replace it by the single set.

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- $\mathcal{F}_\mu :=$ the normal closure of $\langle S_\kappa \mid \kappa \in E \cap \mu \rangle$ for any $\mu \leq \delta$.
  - ⇝ Since there are club many $N$ with $\mathcal{A} \in N$, we have $S_\mathcal{A} \in \mathcal{F}_{\bar{\kappa}}$ for each m.a.c. in $\mathcal{F}_\kappa$.

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? Is $\mathcal{F}_\mu$ nontrivial? Does $\mathcal{F}_\kappa^+ \cap V[G_\kappa] \subseteq \mathcal{F}_\mu^+$ hold for any $\kappa < \mu \leq \delta$?

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  and, $S_\kappa := \pi_\kappa(\tilde{S}_\kappa)$ in $V[G_{\kappa^+}]$.
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? Is $\mathcal{F}_\mu$ nontrivial? Does $\mathcal{F}_\kappa^+ \cap V[G_\kappa] \subseteq \mathcal{F}_\mu^+$ hold for any $\kappa < \mu \leq \delta$?

⇝ We need $\Delta$'s to take care of these.

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# Putting it all together

The entire construction so far is as follows. For any $\kappa \leq \delta$, let

$$\mathcal{F}_\kappa := \text{the normal closure of } \langle S_\mu \mid \mu \in E \cap \kappa \rangle ,$$

$$\Delta_\kappa := \left\{ A \in \mathcal{P}_{\aleph_1}\kappa \;\middle|\; A \cap \omega_1 \in \bigcap_{\mu \in E \cap A} S_\mu \right\},$$

$$\tilde{S}_\kappa := \left\{ N \prec \mathcal{H}_{\kappa^+}^{V[G_\kappa]} \;\middle|\; \begin{array}{ccc} |N| = \aleph_0, & \Delta_\kappa, \kappa \in N, & N \cap \kappa \in \Delta_\kappa, \\ \forall \mathcal{A} \in N : \mathcal{F}_\kappa & & N \cap \omega_1 \in \bigcup(\mathcal{A} \cap N). \end{array} \right\}$$

Then, in $V[G_{\bar{\kappa}}]$, $S_\kappa := \pi_\kappa(\tilde{S}_\kappa)$.
We have to confirm:

- Each $\mathcal{F}_\kappa$ is nontrivial,
- Coherency: $\mathcal{F}_\kappa^+ \cap V[G_\kappa] \subseteq \mathcal{F}_\mu^+$ for any $\kappa < \mu \leq \delta$.

We slightly modify the above construction and *cheat* to make proof simpler.

# Cheating: Universality of the club filter

⋆ We exploit *the universality of $\mathcal{C}_{\omega_1,X}$* to simplify argument:

## Fact 7 (Burke)

*For any (possibly trivial) filter $\mathcal{F}$ on $\mathcal{P}_{\omega_1}X$ normally generated by $\langle S_\alpha \mid \alpha < \kappa \rangle$, TFAE:*

1. *$\mathcal{F}$ is a nontrivial normal filter.*
2. *$\Delta := \{\, A \in [\kappa]^{\aleph_0} \mid \forall \alpha \in A\ A \cap X \in S_\alpha \,\}$ is stationary and $\mathcal{F}$ is the projection of the club filter restricted to $\Delta$; i.e.*

$$A \in \mathcal{F} \iff \Delta \subseteq_{\mathsf{NS}_{\omega_1,\kappa}} \{\, z \in \mathcal{P}_\mu Y \mid z \cap X \in A \,\}.$$

*We write $\mathcal{F} = \mathcal{F}_{\omega_1,X}(\Delta) = \mathrm{pr}_X(\mathcal{C}_{\omega_1,\kappa} \restriction \Delta)$ for such $\mathcal{F}$.*

! Indeed, Farah [3] essentially showed $\mathcal{F}_\kappa = \mathrm{pr}_{\omega_1}(\mathcal{C}_{\omega_1,\kappa} \restriction \Delta_\kappa)$.

⇝ Rather, we adopt this as *the definition* of $\mathcal{F}_\kappa$!

# Characterisation of $\mathcal{F}_{\kappa,X}(\Delta)$

- ⋆ We characterise $\mathcal{F}_{\kappa,X}(\Delta)$ in terms of elementary submodels.
- First, easy closure argument shows:

## Fact 8

*If $C \subseteq \mathcal{P}_{\omega_1}X$ is club, $X, \mathcal{C} \in N \prec \mathcal{H}_\theta$ where $\theta$ is sufficiently large, then $N \cap X \in C$.*

Then we have the following:

## Lemma 9

*For any stationary $\Delta \subseteq \mathcal{P}_{\omega_1}X$, TFAE:*

1. $A \in \mathcal{F}_{\omega_1}(\Delta)$,
2. *for any $N \prec \mathcal{H}_\theta$, if $\Delta, A, X \in N$ and $N \cap X \in \Delta$ then $N \cap \omega_1 \in A$,*
3. *for club many $N \prec \mathcal{H}_\theta$, if $\Delta, A, X \in N$ and $N \cap X \in \Delta$ then $N \cap \omega_1 \in A$.*

# Our Final Definition

For any $\kappa \leq \delta$, let

$$\Delta_\kappa := \left\{ A \in \mathcal{P}_{\aleph_1}\kappa \;\middle|\; A \cap \omega_1 \in \bigcap_{\mu \in E \cap A} S_\mu \right\},$$

$$\mathcal{F}_\kappa := \mathrm{pr}_{\omega_1}(\mathcal{C}_{\omega_1,\kappa} \upharpoonright \Delta_\kappa),$$

$$\tilde{S}_\kappa := \left\{ N \prec \mathcal{H}_{\kappa^+}^{V[G_\kappa]} \;\middle|\; \begin{array}{c} |N| = \aleph_0, \quad \Delta_\kappa, \kappa \in N, \quad N \cap \kappa \in \Delta_\kappa, \\ \forall \mathcal{A} \in N : \text{m.a.c. of } \mathcal{F}_\kappa \; N \cap \omega_1 \in \bigcup(\mathcal{A} \cap N). \end{array} \right\}$$

Then, in $V[G_{\bar{\kappa}}]$, let $S_\kappa := \pi_\kappa(\tilde{S}_\kappa)$.

- Fact 7 assures $S_\kappa \in \mathcal{F}_{\bar{\kappa}}$.
- Remains to show:
  - Each $\Delta_\kappa$ is stationary in $\mathcal{P}_{\aleph_1}\kappa$,
  - Coherency: $\mathcal{F}_\kappa^+ \cap V[G_\kappa] \subseteq \mathcal{F}_\mu^+$ for any $\kappa < \mu$.

$\rightsquigarrow$ We isolate sufficient condition for for these properties.

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# What is a sufficient condition?

**Definition 10**

$\left\langle \dot{\Delta}_\mu \,\middle|\, \mu_0 \leq \mu \leq \kappa \right\rangle$ is called *Coherent Stationary Sequence* if

1. $\dot{\Delta}_\mu \in V^{\mathbb{P}_\mu}$ and $\Vdash_{\mu_0}$ "$\Delta_{\mu_0}$ : stationary in $\mathcal{P}_{\omega_1}\mu_0$",
2. (Monotonicity) $\Vdash_\mu$ "$\Delta_\mu \restriction \nu \subseteq \Delta_\nu$" for any $\nu < \mu \leq \kappa$,
3. (Extension) For any $\nu < \mu \leq \kappa$, the following holds in $V$: Suppose $N \prec \mathcal{H}^V_\theta$, $p \in \mathbb{P}_\mu \cap N$, $q$ is $(N, \mathbb{P}_\nu)$-generic, $p \parallel q$ and $q \Vdash_\nu$ "$N \cap \nu \in \Delta_\nu$". Then, there are $N^* \succ_{\omega_1} N$ and $(N^*, \mathbb{P}_\mu)$-generic $r \leq p, q$ such that $r \Vdash_\mu$ "$N^* \cap \mu \in \Delta_\mu$".

- Here, $N \prec_\lambda N^*$ means $N \prec N^*$ and $N \cap \lambda = N^* \cap \lambda$.
  - Existing proofs require $N^*$ and $N$ to coincide up to $\nu$, but we find that $\omega_1$ is sufficient.
- The last condition needs more explanation.

# Elementary Submodels and Generic Condition

⋆ Extension Property discusses on *generic conditions*:

**Definition 11**

Let $\mathbb{P}$ be a poset $\mathbb{P}$, $\theta$ sufficiently large and $\mathbb{P} \in N \prec \mathcal{H}_\theta$. $p \in \mathbb{P}$ is *$(N, \mathbb{P})$-generic* (or, master) if $p \Vdash$ "$\check{N}[\dot{G}] \cap \mathrm{On} = \check{N} \cap \mathrm{On}$".

**Fact 12 (Shelah [7])**

*The following are equivalent:*

1. $p$ *is* $(N, \mathbb{P})$*-generic,*
2. $p \Vdash$ "$(N[G], N, G, <) \prec (\mathcal{H}_\theta[G], \mathcal{H}_\theta^V, G, <)$",
3. $p \Vdash$ "$N[G] \cap V = N$".

**Remark**

Every $N \prec (\mathcal{H}_\theta[G], \mathcal{H}_\theta^V, G, <)$ can be written as $N = N_0[G]$ for some $N \prec \mathcal{H}_\theta^V$ and $N_0 \cap V = N$.

# Extension of Generic Condition

Classical results on genericity and properness:

**Definition 13**

A forcing notion $\mathbb{P}$ is *proper* if for any countable $N \prec \mathbb{P}$, if $\mathbb{P}$ and $p \in N \cap \mathbb{P}$, then there is $(N, \mathbb{P})$-generic $q \leq p$.

**Fact 14**

- Col$(\lambda, < \kappa)$ *is proper if* $\lambda \geq \omega_1$.
- $\mathbb{P}$ *is proper iff it preserves every stationary* $S \subseteq \mathcal{P}_{\aleph_1} X$.

The following illustrates that Extension Property is a strengthening of properness, which requires $\mathcal{F}_\mu$-positives to be preserved:

**Lemma 15**

*If* $\vec{\Delta}$ *is c.s.s and* $\nu < \mu$, *then, in* $V[G_\mu]$, $\mathcal{F}_\nu^+ \cap V[G_\nu] \subseteq \mathcal{F}_\mu^+$.

Proof

# Use of Extension Property: Stationarity

Stationarity of each $\Delta_\mu$'s can be similarly proven:

**Lemma 16**

*Let $\langle \Delta_\mu \mid \mu_0 \leq \mu \leq \kappa \rangle$ be c.s.s. Then $\Vdash_\mu$ "$\Delta_\mu$ : stationary in $\mathcal{P}_{\omega_1}\mu$" for any $\mu \in E$.*

**Proof.**

Almost the same of coherency of positive sets, but much easier because we don't have to take care of $N \cap \omega_1$. □

# The correctness of our $\Delta$'s and Reflection Principle

⋆ So it remains to show that our $\vec{\Delta}$ is indeed coherent:

**Lemma 17**

*Our definition of $\Delta_\mu$ satisfies the definition of c.s.s.*

- All conditions trivially hold, except for Extension Property.
- Here, a kind of Reflection Principle plays a crucial role:

**Definition 18 (Positive-set Reflection Principle)**

Let $\Delta \subseteq \mathcal{P}_{\omega_1}\kappa$ and $\lambda < \kappa$. The Positive-set Reflection Principle, $\mathsf{PRP}_{\omega_1}(\Delta)$, is the following assertion:
For any sufficiently large $\theta$ and stationary $S \subseteq \{ N \prec \mathcal{H}_\theta \mid N \cap \kappa \in \Delta \}$, there is a continuous $\in$-elementary chain $\langle N_\alpha \prec \mathcal{H}_\theta \mid \alpha < \omega_1 \rangle$ with $\{ \alpha < \omega_1 \mid N_\alpha \in S \} \in \mathcal{F}_{\omega_1}(\Delta)^+$

⇝ $\mathsf{PRP}_{\omega_1}(\Delta)$ implies the classical Stationary Reflection Principle restricted to $\Delta$ if $\omega_1^\omega = \omega_1$.

# PRP for coherent sequence

## Theorem 19

*Let $\langle \Delta_\alpha \mid \alpha \leq \kappa \rangle$ be c.s.s. and $\kappa$ be $2^\kappa$-supercompact. Then* $\mathrm{PRP}_{\omega_1}(\Delta_\kappa)$ *holds.*

## Sketch of Proof.

- Using $2^\kappa$-s.c. embedding $j$ with c.p. $\kappa$, we argue as standard stationary reflection.
- In particular, we can divide $\tilde{H} := j\,``\,\mathcal{H}_{\kappa^+}^{V[G]}$ into $\omega_1$-chain and project $j(\Delta_\kappa)$ along it $T := \{\, \alpha < \omega_1 \mid N_\alpha^* \in j(\Delta_\kappa) \,\}$.
- $T$ sits in $M^{\kappa^{+I}}$ by closure, and it behaves well up to $\kappa^{+I}$; then we use Extension Property in $M$ to lift it up to $j(\kappa)$.

□

▸ Proof

# The Proof of Extension Property

We are now at the point that we can prove the EP of $\Delta_\kappa$'s, i.e:

**Lemma 20**

*Let $\mu < \kappa \in \mathrm{Cl}(E)$. Suppose, in $V$, $N \prec \mathcal{H}_\theta^V$, $p \in \mathbb{P}_\kappa \cap N$, $q$ is $(N, \mathbb{P}_\mu)$-generic, $p \parallel q$ and $q \Vdash_\mu$ "$N \cap \mu \in \Delta_\mu$". Then, there are $N^* \succ N$ and $(N^*, \mathbb{P}_\kappa)$-generic $r \leq p, q$ such that $N^* \cap \omega_1 = N \cap \omega_1$ and $r \Vdash_\kappa$ "$N^* \cap \kappa \in \Delta_\kappa$".*

- Although the range of $\kappa$ is restricted to $\mathrm{Cl}(E)$, it poses no difficulty, since $E$ is stationary.
- Prove this by induction on $(\kappa, \mu)$, divided into three cases:
  a. Successor step: $\kappa = \mu^{+E}$ - we use PRP here,
  b. Essentially successor step: $\kappa > \mu^{+E}$, but $\kappa^* := \sup(E \cap \kappa) < \kappa$. In this case, we use I.H. to extend $p, q$ to $\mathbb{P}_{\kappa^*}$-generic, and then it trivially extends to $\mathbb{P}_\kappa$-generic, since there is no s.c's in-between.
  c. Limit Step: $\kappa > \mu^{+E}$ and $\kappa = \sup(E \cap \kappa)$.

# Reflection Principle and Successor step

Clearly, the successor step is reduced to the following:

**Lemma 21**

*Let $\kappa$ be $2^\kappa$-s.c. and EP hold up to $\kappa$. In $V[G_\kappa]$, if $N \prec \mathcal{H}_\theta$ is such that $N \cap \kappa \in \Delta_\kappa$, then there is $N^* \succ_{\omega_1} N$ with $N^* \cap \kappa \in \Delta_\kappa$ and $N^* \cap H_{\mu^+} \in \tilde{S}_\kappa$.*

Which is obtained by easy bookkeeping argument, repeatedly applying the following:

**Lemma 22 (One-step lemma)**

*Let $\kappa$ be $2^\kappa$-s.c. and EP hold up to $\kappa$. In $V[G_\kappa]$, suppose $\mathcal{A} \in N \prec \mathcal{H}_\theta[G_\kappa]$ is a m.a.c. in $\mathcal{F}^+_\kappa$ and $N \cap \kappa \in \Delta_\kappa$. Then, there is some $N^* \succ_{\omega_1} N$ with $N^* \cap \omega_1 \in \bigcup(N^* \cap \mathcal{A})$ and $N^* \cap \kappa \in \Delta_\kappa$.*

# Proof of One-step Lemma from PRP

Proof. In view of 8, it suffices to show that $T := \{ N \prec \mathcal{H}_{\kappa^+} \mid N \cap \kappa \in \Delta_\kappa, \mathcal{F}_\kappa, \mathcal{A} \in N \}$ is contained in, modulo club, the following[3]:

$$\nabla(\mathcal{A}) := \left\{ N \prec \mathcal{H}_{\kappa^+} \;\middle|\; \exists a \in \mathcal{A} \begin{bmatrix} N^* := \mathrm{Sk}(N \cup \{ a \}) \succ_{\omega_1} N, \\ N^* \cap \omega_1 \in a, N^* \cap \kappa \in \Delta_\kappa \end{bmatrix} \right\}.$$

To see that, we fix arbitrary stationary $A \subseteq T$ and show $A \cap \nabla(\mathcal{A}) \neq \emptyset$. By assumption, we can use $\mathrm{PRP}_{\omega_1}(\Delta_\kappa)$ for $A$; so pick continuous $\in$-elementary chain $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ of $\mathcal{H}_{\kappa^+}$ such that $Z := \{ \alpha < \omega_1 \mid N_\alpha \in A \} \in \mathcal{F}_\kappa^+$. By the definition of $\mathcal{F}_\kappa$, we also have $D := \{ N \cap \omega_1 \mid N \cap \kappa \in \Delta_\kappa, N \prec \mathcal{H}_\theta \} \in \mathcal{F}_\kappa$. Since $\mathcal{A}$ is a m.a.c. in $\mathcal{F}_\kappa^+$, we can pick $a \in \mathcal{A}$ with $a \cap D \cap Z \in \mathcal{F}^+$.

[3] To be more rigorous, we have to use "Catching-your-tails" argument.

# Proof of One-Step Lemma (cont'd)

Hence, we can pick $N_0^* \prec \mathcal{H}_\theta$ such that:

1. $\kappa, A, \mathcal{A}, \vec{N} \in N_0^*$,
2. $\alpha := N_0^* \cap \omega_1 \in a \cap D \cap Z$, and
3. $N_0^* \cap \kappa \in \Delta_\kappa$.

Then $N := N_\alpha$ is as desired. Indeed, $N^* := N \cap \mathcal{H}_{\kappa^+}$ is $\omega_1$-extension of $N$ witnessing $N \in A \cap \nabla(\mathcal{A})$. □

# Easy Sketch for Limit Step

The Limit Step is essentially showed by repeating successor step for countably-many times. In particular, it is enough to construct $\langle N_n, p_n, q_n, \mu_n \mid n < \omega \rangle$ with:

1. $N = N_0 \prec_\lambda N_1 \prec_\lambda N_2 \prec_\lambda \dots$,
2. $\mu_n \nearrow \kappa$ if $\mathrm{cf}(\kappa) = \omega$; $\mathrm{dom}(p_n) \subseteq \mu_{n+1}$ otherwise,
3. $\mu = \mu_0 < \mu_1 < \dots$, $\kappa_n < \mu_{n+1} \in N_n \cap \kappa \cap E$,
4. $q = q_0 \geq q_1 \geq q_2, \dots$, $q_n$: $(N_n, \mathbb{P}_{\kappa_n})$-generic, $q_{n+1} \leq p_n \restriction \mu_{n+1}$, and $q_n \Vdash N_n \cap \mu_n \in \Delta_{\mu_n}$, and
5. $p = p_0 \geq p_1 \geq p_2, \dots$, $p_{n+1} \in D_n \cap N_{n+1}$ and $p_n \parallel q_n$.

Then, $r := \bigcup_n q_n$ will be as desired. The case-splitting on $\mathrm{cf}(\kappa)$ is needed to ensure $r \leq p, q$ by fusion argument. □

# Summary

- We add stationary sets to the club filter, ensuring each m.a.c. $\mathcal{A}$ is added at some intermediate stage.
- This is done by combinatorics of elementary submodels and collapsed onto $\omega_1$ by Lévy collapse.
- We adopt the characterisation exploiting the universality of club filter, which reduces some burden of proof:
  1. Nontriviality of the resulting filter is almost trivial.
  2. Sets like $\{ N \cap \omega_1 \mid N \cap \kappa \in \Delta_\kappa \}$ is easily shown to be measure one.
- We formulate abstract concept of *coherent stationary sequence*, which admits coherency of positive sets and a kind of Reflection Principle.

# Table of Contents

# Construction on $\mathcal{P}_{\omega_1}\lambda$

The result generalises to the following, formerly unmentioned one:

**Theorem 23 (I.)**

*Let $\delta$ be s.c, $\lambda < \delta$ regular, and $G$ a $\mathrm{Col}(\lambda, {<}\delta)$-generic filter over $V$. Then, in $V[G]$, there is a $\lambda^+$-saturated filter on $\mathcal{P}_{\omega_1}\lambda$.*

- Use $\prec_\lambda$-extension instead of $\prec_{\omega_1}$-extension.
- Instead of $\in$-chain, we use *continuous directed systems* of elementary substructures; i.e. $\langle N_x \mid x \in \mathcal{P}_{\omega_1}\kappa\rangle$ s.t.

$$N_x = \bigcup_{z \in [x]^{<\omega}} N_z \text{ (if } |x| \geq \aleph_0), \quad x \subseteq N_x \prec N_y \prec \mathcal{H} \text{ if } x \subseteq y.$$

- We have, for club many $x \in \mathcal{P}_{\omega_1}\kappa$, $N_x \cap \kappa = x$.
- $\mathrm{PRP}_\lambda(\Delta)$ can be similarly formulated and proven.

# Table of Contents

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- ? Is there any other application of this construction?

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# Thank you for your attention!

# Appendix: Detailed Proof

# A Proof of Indestructibility Lemma

- Since $\mathcal{A} \in V[G_\kappa]$, we can list $\mathcal{A} = \{\, f(\alpha) \mid \alpha < \omega_1 \,\}$.
- $D_0 := \{\, N^\kappa_\alpha \mid N^\kappa_\alpha \cap \omega_1 = \alpha, f[\alpha] = \mathcal{A} \cap N^\kappa_\alpha \,\}$ contains a club, hence $D_0 \in \mathcal{F}$ by normality.
  - $\rightsquigarrow$ $D := D_0 \cap \pi_\kappa(\tilde{S}_{\mathcal{A}}) \in \mathcal{F}$.

⋆ In view of Fact 4, it suffices to see $D \subseteq \bigtriangledown_\alpha f(\alpha)$.

- So take $\alpha \in D$; we will show $\alpha \in \bigcup_{\gamma<\alpha} f(\gamma)$.
- Since $\alpha \in D_0$, we have $N^\kappa_\alpha \cap \alpha = \alpha$ and $f[\alpha] = \mathcal{A} \cap N^\kappa_\alpha$.
- On the other hand. $\alpha \in \pi_\kappa(\tilde{S}_{\mathcal{A}})$ implies we have $\alpha = N^\kappa_\alpha \cap \omega_1 \in \bigcup(\mathcal{A} \cap N^\kappa_\alpha)$.

$\rightsquigarrow$ $\alpha \in \bigcup_{\gamma<\alpha} f(\gamma)$ as desired! □

Back

# Use of Extension Property: Positive-Set Coherency

Proof. Fix $\dot{A}$ with $\Vdash_\nu$ "$\dot{A} \in \mathcal{F}_\nu^+$". In view of Lemma 9, it means

$$\Vdash_\nu \text{“}\left\{ N \prec \mathcal{H}_\theta \,\middle|\, N \cap \nu \in \Delta_\nu, N \cap \omega_1 \in \dot{A} \right\} : \text{ stationary”.}$$

We fix any $p \in \mathbb{P}_\mu$ and find $r \leq p$ with $r \Vdash_\mu \dot{A} \in \mathcal{F}_\mu^+$. Again, by Lemma 9, it suffices to find $N^* \prec \mathcal{H}_\theta^V$ and $(N^*, \mathbb{P}_\mu)$-generic $r \leq p$ such that:

$$r \Vdash N^* \cap \mu \in \Delta_\mu \wedge N^* \cap \omega_1 \in \dot{A} \wedge \Delta_\mu, A, \mu \in N^*.$$

Recall that $N^*$-genericity assures that $N^*$ and $N^*[G_\kappa]$ has exactly the same ordinals. By stationarity, we can pick $\dot{N} \in V^{\mathbb{P}_\mu}$ such that:

$$p \Vdash_\mu \text{“}\dot{N}[G_\kappa] \prec \mathcal{H}_\theta[G_\kappa], \dot{N}[G_\nu] \cap \omega_1 \in \dot{A}, \dot{N}[G_\nu] \cap \nu \in \Delta_\nu,$$
$$p, \nu, \mu, \dot{A}, \dot{\Delta}_\nu, \check{\dot{\Delta}}_\mu \in N[G_\nu]\text{”.}$$

# Proof of Coherency of Positive Sets (cont'd)

Since $\mathbb{P}_\mu$ is countably closed, we can pick $q_0 \leq p$ and $N \prec \mathcal{H}_\theta^V$ such that $q_0 \Vdash \dot{N} = \check{N}$.
Let $q := q_0 \restriction \nu$. Then $q$ is $(N, \mathbb{P}_\nu)$-generic. Furthermore, since the statement $\check{N} \cap \nu \in \dot{\Delta}_\nu$ is determined at $\nu$-stage, we have $q \Vdash$ "$\check{N} \cap \nu \in \dot{\Delta}_\nu$". By definition we also have $q \parallel p$.
Then, EP gives us $N^* \succ N$ and $(N^*, \mathbb{P}_\mu)$-generic $r \leq p, q$ with

$$r \Vdash \text{“}\dot{A}, \dot{\Delta}_\mu, \mu \in N^*[G_\mu] \wedge N^* \cap \mu \in \dot{\Delta}_\mu \wedge N^* \cap \omega_1 = N \cap \omega_1 \in \dot{A}\text{”},$$

which is what we wanted. □

▸ Back

# Proof of PRP for a coherent sequence

Proof. It suffices to show the case $\theta = \kappa^+$. First we fix an $2^\kappa$-s.c. embedding $j : V \overset{\prec}{\longrightarrow} M$ with $\mathrm{cp}(j) = \kappa$. Since ${}^{2^\kappa}M \subseteq M$, we have $\mathcal{H}^V_{\kappa^+} = \mathcal{H}^M_{\kappa^+}$; in particular, we have $\mathcal{H}^{V[G_\kappa]}_{\kappa^+} = \mathcal{H}^{M[G_\kappa]}_{\kappa^+}$ for any $(V, \mathbb{P}_\kappa)$-generic $G_\kappa$. Further we have $j \restriction \mathcal{H}_{\kappa^+} \in M$. So let, in $M^{\mathbb{P}_{j(\kappa)}}$, $N^*_\alpha := j(N^\kappa_\alpha) \prec \mathcal{H}^{M[\dot{K}]}_{j(\kappa)^+}$ and $\tilde{H} := \bigcup_\alpha N^*_\alpha$.

Fix any $\dot{S}$ such that $\Vdash^V_\kappa$ "$\dot{S} \subseteq \left\{ N \prec \mathcal{H}^+_\kappa[\dot{G}] \;\middle|\; N \cap \kappa \in \Delta_\kappa \right\}$". By elementarity, it suffices to show the following:

**Claim**

$$\Vdash^M_{j(\delta)} \dot{B} := \left\{ \alpha < \omega_1 \;\middle|\; N^*_\alpha \in j(\dot{S}) \right\} \in \mathcal{F}_{\omega_1}(j(\Delta)_{j(\kappa)})^+.$$

# Proof of PRP for a coherent sequence (cont'd)

So we will argue in $M$.

Note that, again by closure, we have $\dot{S} \in M$. Hence, by elementarity, we have $\Vdash^{M}_{j(\kappa)} \dot{B} = \pi_\kappa(\dot{S})$, which means that $\dot{B}$ is stationary in $M^{j(\kappa)}$ and we may assume that $\dot{B} \in M^{\kappa^+}$.

With these and Lemma 9 in mind, the above reduces to the following:

**Claim'**

For any $p \in \mathbb{P}_{j(\kappa)}$, there is $N^* \prec \mathcal{H}^M_{j(\theta)}$ and $(N^*, \mathbb{P}_{j(\kappa)})$-generic $r \leq p$ which forces $j(\kappa), j(\Delta)_{j(\kappa)} \in N^*[G_{j(\kappa)}]$, $N^* \cap j(\kappa) \in j(\Delta)_{j(\kappa)}$ and $N^* \cap \omega_1 \in \dot{B}$.

So fix any $p \in \mathbb{P}_{j(\delta)}$.

# Proof of PRP: Taking generic $r$

Since $\dot{B}$ is stationary, one can pick $\dot{N} \in M^{j(\kappa)}$ such that

$$p \Vdash \check{p}, \kappa^+, j(\kappa), \Delta_{\kappa^+}, \Delta_{j(\kappa)} \in \dot{N}[G_{j(\kappa)}] \prec \mathcal{H}_{j(\kappa)}[G_{j(\kappa)}], \dot{N} \cap \omega_1 \in \dot{B}.$$

Take $q_0 \leq p$ and $N \prec \mathcal{H}^M_{j(\kappa)}$ such that $q_0 \Vdash \dot{N} = \check{N}$ and let $q := q \restriction \kappa^+$ and $\alpha := N \cap \omega_1$. We may assume that $N^\kappa_\alpha \cap \omega_1 = N^* \cap \omega_1 = \alpha$ and clearly $q$ is $(N, \mathbb{P}_{\kappa^+})$-generic. Then we have $q_0 \Vdash \alpha \in \dot{B}$, and hence $q \Vdash$ "$N \cap \omega_1 \in \dot{B}$". We also have $q_0 \Vdash N^\kappa_\alpha \cap j(\kappa) \in j(\Delta)_{j(\kappa)}$. But, since $N^\kappa_\alpha = N \cap \tilde{H}$ and $\tilde{H} \cap j(\kappa) = \tilde{H} \cap \kappa^{+I} = \kappa$, we have $N \cap \kappa^+ \in j(\Delta)_{j(\kappa)}$. In particular, Monotonicity of $\vec{\Delta}$ implies that $N \cap \kappa^+ \in j(\Delta)_{\kappa^+}$. Then, by Extension Property, we can get $N^* \succ N$ and $(N^*, \mathbb{P}_{j(\kappa)})$-generic $r \leq p, q$ such that $N^* \cap \omega_1 = \alpha$ and $r \Vdash$ "$N^* \cap j(\kappa) \in j(\Delta)_{j(\kappa)} \wedge N^* \cap \omega_1 \in \dot{B}$", which was what we wanted. $\square$

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# $\lambda$-SR from PRP

**Lemma 24**

*Suppose* $\operatorname{cf}\omega_1 \leq \lambda < \kappa$, $\lambda^{<\omega_1} = \lambda$ *and* $\Delta$ *is weakly stationary in* $\mathcal{P}_{\omega_1}\kappa$. *If* $\mathsf{PRP}_{\omega_1}(\Delta, \lambda)$ *holds, then, for any* $S \subseteq \Delta$ *weakly stationary in* $\mathcal{P}_{\omega_1}\kappa$, *there is* $X \in [\kappa]^{\lambda^{<\omega_1}}$ *such that* $\lambda \subseteq X$ *and* $S \cap \mathcal{P}_{\omega_1}X$ *remains weakly stationary in* $\mathcal{P}_{\omega_1}X$.

Proof. Let $S \subseteq \mathcal{P}_{\omega_1}\kappa$ be stationary. Fix sufficiently large $\theta \gg \kappa$.
Clearly, $S^{H_\theta} = \{ N \prec \mathcal{H}_\theta \mid N \cap \kappa \in S \}$ is stationary. By $\mathsf{PRP}_{\omega_1}(\Delta, \lambda)$, there exists a continuous elementary directed system $\langle N_x \mid x \in \mathcal{P}_{\omega_1}\lambda \rangle$ such that $T := \{ x \in \mathcal{P}_{\omega_1}\lambda \mid N_x \cap \kappa \in S \}$ is $\mathcal{F}_\Delta$-positive, and, in particular, stationary.
Let $H := \bigcup_x N_x$ and $X := H \cap \lambda$. Then we have $|X| = \lambda^{<\omega_1}$ and clearly $\lambda \subseteq X$. We claim that this $X$ suffice.

# $\lambda$ – SR from PRP (cont'd)

Lifting $T$ up to $\mathcal{P}_{\omega_1}X$, we have that $T^X := \{\, z \in \mathcal{P}_{\omega_1}X \mid N_{z\cap\lambda} \cap \kappa \in S \,\}$ is stationary. It suffices to show that $C := \{\, z \in \mathcal{P}_{\omega_1}X \mid N_{z\cap\lambda} \cap \kappa = z \,\}$ contains club, since it implies that $T^X \subseteq_{\mathcal{C}_{\omega_1,X}} S$, and hence $S \cap \mathcal{P}_{\omega_1}X$ is stationary as desired.

To see that, let $D := \{\, N_x \cap X \mid x \in \mathcal{P}_{\omega_1}\lambda, N_x \cap \lambda = x \,\}$, which is club in $\mathcal{P}_{\omega_1}X$. We have $D \subseteq C$: if $z \in D$, then, $z = N_x \cap X$ for some $x \in \mathcal{P}_{\omega_1}\lambda$, and by definition of $D$ we have $z \cap \lambda = x$. It follows that $z = N_x \cap \kappa = N_{z\cap\lambda} \cap \kappa$.